Week 2 - Wednesday

COMP 2230

Last time

- More on implications
- Arguments

Questions?

Assignment 1

Logical warmup

- A group of airplanes is based on a small island.
- The tank of each plane holds just enough fuel to take it halfway around the world.
- Any amount of fuel can be transferred from the tank of one plane to the tank of another while the planes are in flight.
- The only source of fuel is on the island, and for the purpose of the problem it is assumed that there is no time lost in refueling either in the air or on the ground.
- What is the smallest number of planes that will ensure the flight of one plane around the world on a great circle, assuming that the planes have the same constant ground speed and rate of fuel consumption and that all planes return safely to their island base?

Predicate Logic

Predicates

- A **predicate** is a sentence with a fixed number of variables that becomes a **statement** when specific values are substituted for to the variables
- The **domain** gives all the possible values that can be substituted
- The truth set of a predicate $P(x)$ are those elements of the domain that make $P(x)$ true when they are substituted

Predicate examples

- Let $P(x)$ be "x has had 4 wisdom teeth removed"
- What is the truth set if the domain is the people in this classroom?

- Let $Q(n)$ be "n is divisible by exactly itself and 1"
- What is the truth set if the domain is the set of positive integers $\mathbb{Z}^+$?

A note about sets

- We will frequently be referring to various sets of numbers in this class
- Some typical notation used for these sets:

Symbol	Set	Examples
$\mathbb{R}$	Real numbers	Virtually everything that isn't imaginary
$\mathbb{Z}$	Integers	{…, -2, -1, 0, 1, 2,…}
$\mathbb{Z}^-$	Negative integers	{-1, -2, -3, …}
$\mathbb{Z}^+$	Positive integers	{1, 2, 3, …}
$\mathbb{N}$	Natural numbers	{1, 2, 3, …}
$\mathbb{Q}$	Rational numbers	a/b where $a, b \in \mathbb{Z}$ and $b \neq 0$

- Some authors use $\mathbf{Z}^+$ to refer to non-negative integers and only **N** for the natural numbers

Universal quantification

- The universal quantifier $\forall$ means "for all"
- The statement "All DJs are mad ill" can be written more formally as:
- $\forall x \in D, M(x)$
  - Where D is the set of DJs and $M(x)$ denotes that x is mad ill

Universal quantification examples

- Let $S = \{1, 2, 3, 4, 5\}$
- Show that the following statement is true:
  - $\forall x \in S, x^2 \geq x$

- Show that the following statement is false:
  - $\forall x \in \mathbb{R}, x^2 \geq x$

Existential quantification

- The universal quantifier $\exists$ means "there exists"
- The statement "Some emcee can bust a rhyme" can be written more formally as:
- $\exists y \in E, B(y)$
  - Where E is the set of emcees and $B(y)$ denotes that y can bust a rhyme

Existential quantification examples

- Let $S = \{2, 4, 6, 8\}$
- Show that the following statement is false:
  - $\exists x \in S, 1/x = x$

- Show that the following statement is true:
  - $\exists\, x \in \mathbb{Z}, 1/x = x$

More quantified examples

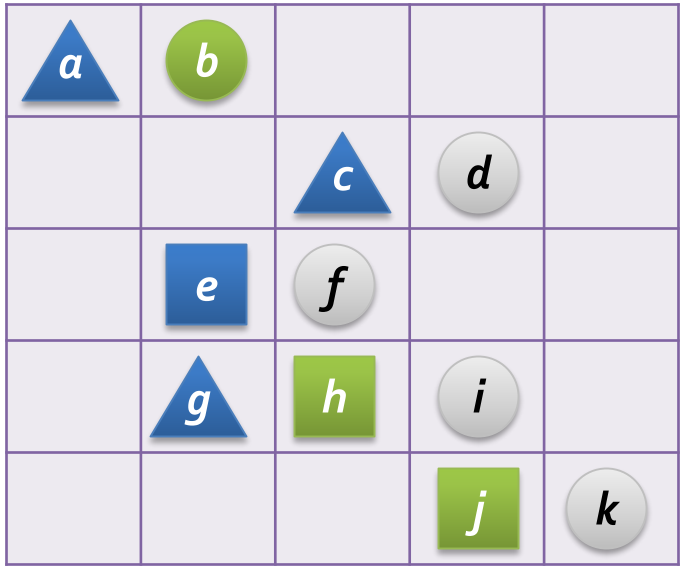
- Convert the following statements in English into quantified statements of predicate logic
- The set P is the set of all people
  - Every son is a descendant
  - Every person is a son or a daughter
  - There is someone who is not a descendant
  - Every parent is a son or a daughter
  - There is a descendant who is not a son

Tarski's World

- Tarski's World provides an easy framework for testing knowledge of quantifiers
- The following notation is used:
  - Triangle(x) means "x is a triangle"
  - Blue(y) means "y is blue"
  - RightOf(x, y) means "x is to the right of y (but not necessarily on the same row)"

Tarski's World Example

- Are the following statements true or false?
  - $\forall t$, Triangle$(t) \rightarrow$ Blue(t)
  - $\forall x$, Blue$(x) \rightarrow$ Triangle(x)
  - $\exists y$ such that Square$(y) \wedge$ RightOf(d, y)
  - $\exists$z such that Square$(z) \wedge$ Gray(z)



Negating Quantifiers and Multiple Quantifiers

Negating quantified statements

- When doing a negation, negate the predicate and change the universal quantifier to existential or vice versa
- Formally:
  - $\sim(\forall x, P(x)) \equiv \exists x, \sim P(x)$
  - $\sim(\exists x, P(x)) \equiv \forall x, \sim P(x)$
- Thus, the negation of "Every dragon breathes fire" is "There is one dragon that does not breathe fire"

Negation example

- Argue the following:
  - "Every unicorn has five legs"
- First, let's write the statement formally
  - Let $U(x)$ be "x is a unicorn"
  - Let $F(x)$ be "x has five legs"
  - $\forall x, U(x) \rightarrow F(x)$
- Its negation is $\exists x, \sim(U(x) \rightarrow F(x))$
  - We can rewrite this as $\exists x, U(x) \wedge \sim F(x)$
- Informally, this is "There is a unicorn which does not have five legs"
- Clearly, this is false
- If the negation is false, the statement must be true

Vacuously true

- The previous slide gives an example of a statement which is **vacuously true**
- When we talk about "all things" and there's nothing there, we can say anything we want

Multiple Quantifiers

Multiple quantifiers

- So far, we have not had too much trouble converting informal statements of predicate logic into formal statements and vice versa
- Many statements with multiple quantifiers in formal statements can be ambiguous in English
- Example:
  - "There is a person supervising every detail of the production process."

Example

- "There is a person supervising every detail of the production process."
- What are the two ways that this could be written formally?
  - Let D be the set of all details of the production process
  - Let P be the set of all people
  - Let $S(x, y)$ mean "x supervises y"

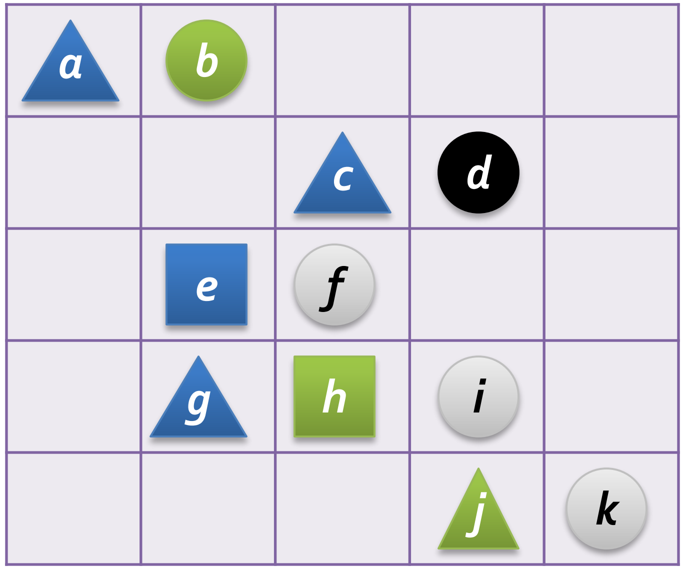
- $\forall x \in D, \exists y \in P$ such that $S(x, y)$

- $\exists y \in P, \forall x \in D$ such that $S(x, y)$

Mechanics

- Intuitively, we imagine that corresponding "actions" happen in the same order as the quantifiers
- The action for $\forall x \in A$ is something like, "pick any x from A you want"
- Since a "for all" must work on everything, it doesn't matter which you pick
- The action for $\exists y \in B$ is something like, "find some y from B"
- Since a "there exists" only needs one to work, you should try to find the one that matches

Tarski's World Example



- Is the following statement true?
- "For all blue items x, there is a green item y with the same shape."
- Write the statement formally.
- Reverse the order of the quantifiers. Does its truth value change?

Practice

- Given the formal statements with multiple quantifiers for each of the following:
  - There is someone for everyone.
  - All roads lead to some city.
  - Someone in this class is smarter than everyone else.
  - There is no largest prime number.

Negating multiply quantified statements

- The rules don't change
- Simply switch every ∀ to ∃ and every ∃ to ∀
- Then negate the predicate
- Write the following formally:
  - "Every rose has a thorn"
- Now, negate the formal version
- Convert the formal version back to informal

Changing quantifier order

- As show before, changing the order of quantifiers can change the truth of the whole statement
- However, it does not necessarily
- Furthermore, quantifiers of the same type are commutative:
  - You can reorder a sequence of $\forall$ quantifiers however you want
  - The same goes for $\exists$
  - Once they start overlapping, however, you can't be sure anymore

Ticket Out the Door

Upcoming

Next time…

- Arguments with predicates
- Basic proofs and counterexamples

Reminders

- Read Sections 3.4, 4.1, and 4.2
- Keep working on Assignment 1